

COMPACTIFIED PICARD STACKS OVER $\overline{\mathcal{M}}_g$

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ABSTRACT. We study algebraic (Artin) stacks over $\overline{\mathcal{M}}_g$ giving a functorial way of compactifying the relative degree d Picard variety for families of stable curves. We also describe for every d the locus of genus g stable curves over which we get Deligne-Mumford stacks strongly representable over $\overline{\mathcal{M}}_g$.

1. INTRODUCTION

In this paper we study (compactified) moduli stacks of line bundles over the moduli stack of stable curves $\overline{\mathcal{M}}_g$. The aim is to give a functorial way of getting compactified Picard varieties (of degree d) for families of stable curves. In other words, we study geometrically meaningful algebraic stacks $\overline{\mathcal{P}}_{d,g}$ with a map to $\overline{\mathcal{M}}_g$ such that, given a family of stable curves $f : \mathcal{X} \rightarrow S$, the fiber product of $\overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ by the moduli map $\mu_f : S \rightarrow \overline{\mathcal{M}}_g$ is either a compactification of the relative degree d Picard variety associated to f or has a canonical map onto it (see below).

There are many constructions of compactified Picard varieties of stable curves. We will choose the one built by Caporaso in [C94]. This compactification, $\overline{\mathcal{P}}_{d,g}$, is constructed as a GIT-quotient and has a proper morphism $\phi_d : \overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ such that $\phi_d^{-1}(M_g^0)$ is isomorphic to the “universal Picard variety of degree d ”, Pic_g^d , parametrize isomorphism classes of line bundles of degree d over automorphism-free nonsingular curves. Points in $\overline{\mathcal{P}}_{d,g}$ correspond to isomorphism classes of balanced line bundles of degree d in quasistable curves of genus g (see Definition 1.1). In particular, given $[X] \in \overline{\mathcal{M}}_g^0$, the smooth locus of $\phi_d^{-1}(X)$ is isomorphic to the disjoint union of a finite number of copies of the jacobian of X , J_X .

Let $f : \mathcal{X} \rightarrow S$ be a family of (genus g) stable curves. By a compactification of the relative Picard variety of degree d associated to f we mean a projective S -scheme P whose fiber over closed points ξ of S is isomorphic to $\phi_d^{-1}(X_\xi)$ (X_ξ denotes the fiber of f over ξ).

Let $(d - g + 1, 2g - 2) = 1$. Then, the GIT-quotient yielding $\overline{\mathcal{P}}_{d,g}$ is geometric (see [C94], Prop. 6.2) and the quotient stack associated to it, $\overline{\mathcal{P}}_{d,g}$, is a Deligne-Mumford stack with a strongly representable morphism onto $\overline{\mathcal{M}}_g$ (see [C05], 5.9). So, given a family of stable curves $f : \mathcal{X} \rightarrow S$, the base change of the moduli map $\mu_f : S \rightarrow \overline{\mathcal{M}}_g$ by $\overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ is a scheme, $\overline{\mathcal{P}}_f^d$, yielding a compactification of the relative degree d Picard variety associated to f . Moreover, $\overline{\mathcal{P}}_{d,g}$ parametrizes Néron models of Jacobians in the following way. Let B be a smooth curve defined over an algebraically closed

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field k with function field K and \mathcal{X}_K a smooth genus g curve over K whose regular minimal model over B is a family $f : \mathcal{X} \rightarrow B$ of stable curves. Then, the smooth locus of the map $\overline{\mathcal{P}}_f^d \rightarrow B$ is isomorphic to the Néron model of $\text{Pic}^d \mathcal{X}_k$ over B (see Theorem 6.1 of loc. cit.).

If $(d - g + 1, 2g - 2) \neq 1$, the quotient stack $\overline{\mathcal{P}}_{d,g}$ is not Deligne-Mumford. In particular, its natural map onto $\overline{\mathcal{M}}_g$ is not representable. In section 2 we consider the restriction of $\overline{\mathcal{P}}_{d,g}$ to the locus $\overline{\mathcal{M}}_g^d$ of d -general curves, i.e., the locus of genus g stable curves over which the GIT-quotient above is geometric. In Prop. 2.5 we show that this restriction, denoted by $\overline{\mathcal{P}}_{d,g}^{\text{Ner}}$, is a Deligne-Mumford stack and is endowed with a strongly representable map onto $\overline{\mathcal{M}}_g^d$. So, it gives a functorial way of getting a compactification of the relative degree d Picard variety for families of d -general curves, generalizing Prop. 5.9 of [C05].

In section 2.2 we give a combinatorial description of the locus in $\overline{\mathcal{M}}_g$ of d -general curves, $\overline{\mathcal{M}}_g^d$. For each d , $\overline{\mathcal{M}}_g^d$ is an open subscheme of $\overline{\mathcal{M}}_g$ containing all genus g irreducible curves and $\overline{\mathcal{M}}_g^d = \overline{\mathcal{M}}_g^{d'}$ if and only if $(d - g + 1, 2g - 2) = (d' - g + 1, 2g - 2)$. In Prop. 2.17 we also show that the $\overline{\mathcal{M}}_g^d$ yield a lattice of open subschemes of $\overline{\mathcal{M}}_g$ parametrized by the (positive) divisors of $2g - 2$.

Let us now consider a family $f : \mathcal{X} \rightarrow S$ of genus g stable curves which are not d -general. Then the fiber product of the moduli map of f , $\mu_f : S \rightarrow \overline{\mathcal{M}}_g$ by the map $\overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ is not a scheme. Indeed, it is not even an algebraic space. The best we can get here is that it is canonically endowed with a proper map onto a scheme yielding a compactification of the relative degree d Picard variety associated to f (see Prop. 4.7).

In [C05] 5.10, a modular description of the quotient stack $\overline{\mathcal{P}}_{d,g}$ is given in the case $(d - g + 1, 2g - 2) = 1$. In section 3 we search for a modular description of $\overline{\mathcal{P}}_{d,g}$ for any d . The main difficulty here is that in the general case it is not possible to construct the analogue of Poincaré line bundles for families of stable curves, fundamental in the modular description given in loc. cit, 5.10. To overcome this difficulty we will study the moduli stack of balanced line bundles of relative degree d for families of quasistable curves of genus g , defined in 3.1 and denoted by $\overline{\mathcal{G}}_{d,g}$. In Theorem 3.2 we show that $\overline{\mathcal{G}}_{d,g}$ is an algebraic stack with a morphism onto $\overline{\mathcal{M}}_g$. However, there is an action of \mathbb{G}_m on $\overline{\mathcal{G}}_{d,g}$ compatible with the map to $\overline{\mathcal{M}}_g$, so it cannot be representable over $\overline{\mathcal{M}}_g$. In section 4 we give a modular description of the rigidification (defined by Abramovich, Corti and Vistoli in [ACV01]) of $\overline{\mathcal{G}}_{d,g}$ along the action of \mathbb{G}_m and we show that it is isomorphic to $\overline{\mathcal{P}}_{d,g}$. Moreover, $\overline{\mathcal{G}}_{d,g}$ is a \mathbb{G}_m -gerbe over $\overline{\mathcal{P}}_{d,g}$.

Finally, we ask if, for every d , $\overline{\mathcal{P}}_{d,g}$ parametrizes Néron model of Jacobians of smooth curves as it does if $(d - g + 1, 2g - 2) = 1$.

In the last section we show that the answer is no if $(d - g + 1, 2g - 2) \neq 1$, essentially because $\overline{\mathcal{P}}_{d,g}$ is not representable over $\overline{\mathcal{M}}_g$. We here focus on the case $d = g - 1$, which is particularly important. In fact for this degree all known compactified Jacobians are canonically isomorphic and are endowed with a theta divisor which is Cartier and ample (see [A04]).

1.1. Preliminaries and notation. We will always consider schemes and algebraic stacks locally of finite type over an algebraically closed base field k .

A curve X will always be a connected projective curve over k having at most nodes as singularities. We will denote by C_1, \dots, C_γ the irreducible components of X .

1.1.1. Line bundles on reducible curves. We will denote by ω_X the canonical or dualizing sheaf of X . For each proper subcurve Z of X (which we always assume to be complete), denote by $Z' := \overline{X} \setminus \overline{Z}$, by $k_Z := \sharp(Z \cap Z')$ and by g_Z its arithmetic genus. Recall that, if Z is connected, the adjunction formula gives

$$(1.1) \quad w_Z := \deg_Z \omega_X = 2g_Z - 2 + k_Z.$$

For $L \in \text{Pic} X$ its *multidegree* is $\underline{\deg} L := (\deg_{C_1} L, \dots, \deg_{C_\gamma} L)$ and its (total) degree is $\deg L := \deg_{C_1} L + \dots + \deg_{C_\gamma} L$.

Given $\underline{d} = (d_1, \dots, d_\gamma) \in \mathbb{Z}^\gamma$, we set $\text{Pic}^{\underline{d}} X := \{L \in \text{Pic} X : \underline{\deg} L = \underline{d}\}$ and $\text{Pic}^d X := \{L \in \text{Pic} X : \deg L = d\}$. We have that $\text{Pic}^d X = \sum_{|\underline{d}|=d} \text{Pic}^{\underline{d}} X$, where $|\underline{d}| = \sum_{i=1}^\gamma d_i$.

The *generalized jacobian* of X is

$$\text{Pic}^0 X = \{L \in \text{Pic} X : \underline{\deg} L = (0, \dots, 0)\}.$$

1.1.2. The relative Picard functor. Let X be an S -scheme with structural morphism $\pi : X \rightarrow S$. Given another S -scheme T , we will denote by $\pi_T : X_T \rightarrow T$ the base-change of π under the structural morphism $T \rightarrow S$.

$$\begin{array}{ccc} X_T := T \times_S X & \longrightarrow & X \\ \pi_T \downarrow & & \downarrow \pi \\ T & \longrightarrow & S \end{array}$$

By a family of nodal curves we mean a proper and flat morphism of schemes over k , $f : \mathcal{X} \rightarrow B$, such that every closed fiber of f is a connected nodal curve.

We will denote by $\mathcal{P}ic_f$ the *relative Picard functor* associated to f and by $\mathcal{P}ic_f^d$ its subfunctor of line bundles of relative degree d . $\mathcal{P}ic_f$ is the fppf-sheaf associated to the functor $\mathcal{P} : \text{SCH}_B \rightarrow \text{Sets}$ which associates to a scheme T over B the set $\text{Pic}(\mathcal{X}_T)$. In particular, if the family f has a section, $\mathcal{P}ic_f(T) = \text{Pic}(\mathcal{X}_T)/\text{Pic}(T)$ (see [BLR], chapter 8 for the general theory about the construction of the relative Picard functor).

Thanks to more general results of D. Mumford and A. Grothendieck in [M66] and [Gr], we know that $\mathcal{P}ic_f$ (and also $\mathcal{P}ic_f^d$) is representable by a scheme Pic_f , which is separated if all geometric fibers of f are irreducible (see also [BLR], 8.2, Theorems 1 and 2). Pic_g^d , the “universal degree d Picard variety”, coarsely represents the degree d Picard functor for the universal family of (automorphism-free) nonsingular curves of genus g , $f_g : \mathcal{Z}_g \rightarrow M_g^0$. Furthermore, it was proved by Mestrano and Ramanan in [MR85] for $\text{char } k=0$ and later on by Caporaso in [C94] for any characteristic that Pic_g^d is a fine moduli space, that is, there exists a Poincaré line bundle over

$\text{Pic}_g^d \times_{M_g^0} \mathcal{Z}_g$, if and only if the numerical condition $(d - g + 1, 2g - 2) = 1$ is satisfied.

1.1.3. Stable and semistable curves. A *stable* curve is a nodal connected curve of genus $g \geq 2$ with ample dualizing sheaf. We will denote by \overline{M}_g (resp. $\overline{\mathcal{M}}_g$) the moduli scheme (resp. stack) of stable curves and by $\overline{M}_g^0 \subset \overline{M}_g$ the locus of curves with trivial automorphism group.

A *semistable* curve is a nodal connected curve of genus $g \geq 2$ whose dualizing sheaf has non-negative multidegree.

A nodal curve X is *stable* (resp. *semistable*) if, for every smooth rational component E of X , $k_E \geq 3$ (resp. $k_E \geq 2$.) If X is semistable, the smooth rational components E such that $k_E = 2$ are called *exceptional*.

A semistable curve is called *quasistable* if two exceptional components never meet.

The *stable model* of a semistable curve X is the stable curve obtained by contracting all the exceptional components of X .

A *family of stable* (resp. *semistable*, resp. *quasistable*) curves is a flat projective morphism $f : \mathcal{X} \rightarrow B$ whose geometrical fibers are stable (resp. semistable, resp. quasistable) curves. A line bundle of degree d on such a family is a line bundle on \mathcal{X} whose restriction to each geometric fiber has degree d .

1.2. Balanced line bundles over semistable curves. Recall that Gieseker's construction of \overline{M}_g consists of a GIT-quotient of the action of $PGL(N)$ on a Hilbert scheme where it is possible to embed all semistable curves of genus g (the "Hilbert point" of the curve) (see [G82]). Gieseker shows that in this Hilbert scheme, in order for the Hilbert point of a curve to be GIT-semistable, it is necessary that the multidegree of the line bundle giving its projective realization must satisfy an inequality, called the "Basic Inequality". Later, in [C94], Caporaso shows that this condition is also sufficient.

We will now give the definition of this inequality, extending the terminology introduced in [CCC04].

Definition 1.1. *Let X be a semistable curve of genus $g \geq 2$ and L a degree d line bundle on X .*

- (i) *We say that L (or its multidegree) is semibalanced if, for every connected proper subcurve Z of X the following ("Basic Inequality") holds*

$$(1.2) \quad m_Z(d) := \frac{dw_Z}{2g-2} - \frac{k_Z}{2} \leq \deg_Z L \leq \frac{dw_Z}{2g-2} + \frac{k_Z}{2} := M_Z(d).$$

- (ii) *We say that L (or its multidegree) is balanced if it is semibalanced and if $\deg_E L = 1$ for every exceptional component E of X . The set of balanced line bundles of degree d of a curve X is denoted by B_X^d .*
- (iii) *We say that L (or its multidegree) is stably balanced if it is balanced and if for each connected proper subcurve Z of X such that $\deg_Z L = m_Z(d)$, the complement of Z , Z' , is a union of exceptional components.*

The set of stably balanced line bundles of degree d on X will be denoted by B_X^d .

Remark 1.2. Balanced multidegrees are representatives for multidegree classes of line bundles on X up to twistors (that is, to elements in the degree class group of X , Δ_X , which is a combinatorial invariant of the curve defined in [C94]). More particularly, in [C05], Proposition 4.12, Caporaso shows that, if X is a quasistable curve, every multidegree class in Δ_X has a semibalanced representative and that a balanced multidegree is unique in its equivalence class if and only if it is stably balanced.

We now list some easy consequences of the previous definition.

- Remark 1.3.** (A) If a semistable curve X admits a balanced line bundle L , then X must be quasistable.
- (B) To verify that a line bundle L is balanced it is enough to check that $\deg_Z L \geq m_Z(d)$, for each proper subcurve Z of X and that $\deg_E L = 1$ for each exceptional component E of X .
- (C) If X is a stable curve, then a balanced line bundle L on X is stably balanced if and only if, for each proper connected subcurve Z of X , $\deg_Z L \neq m_Z(d)$.
- (D) Let X be a stable curve consisting of two irreducible components, Z and Z' , meeting in an arbitrary number of nodes. Then X admits a degree d line bundle which is balanced but not stably balanced if and only if $\frac{d-g+1}{2g-2}w_Z \in \mathbb{Z}$ (equivalently if $\frac{d-g+1}{2g-2}w_{Z'} \in \mathbb{Z}$).
- (E) A line bundle is balanced (resp. stably balanced) if and only if $L \otimes \omega_X^{\otimes n}$ is balanced (resp. stably balanced), for $n \in \mathbb{Z}$. So, given integers d and d' such that $\exists n \in \mathbb{Z}$ with $d \pm d' = n(2g-2)$, there are natural isomorphisms $B_X^d \cong B_X^{d'}$ (and $\tilde{B}_X^d \cong \tilde{B}_X^{d'}$).

For (A) and (B) see [CE] Remark 3.3. (C) and (E) are immediate consequences of the definition. For (D) note that, given a balanced γ -uple $\underline{d} \in \mathbb{Z}^\gamma$ such that $|\underline{d}| = d$, there exists a (balanced degree d) line bundle L in X such that $\deg L = \underline{d}$. Since $k_Z = w_Z - 2g_Z + 2$, we can write $m_Z(d)$ as $\frac{d-g+1}{2g-2}w_Z + g_Z - 1$, which is an integer by hypothesis. In the same way $m_{Z'}(d) = d - m_Z(d)$ is an integer too, so $(m_Z(d), m_{Z'}(d))$ is a balanced multidegree which is not stably balanced.

1.3. The compactified Picard variety of degree d over $\overline{\mathcal{M}}_g$. Let $\overline{P}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ be Caporaso's compactification of the universal Picard variety of degree d , $\text{Pic}_g^d \rightarrow M_g^0$, constructed in [C94].

For $d \gg 0$, $\overline{P}_{d,g}$ is the GIT-quotient

$$\pi_d : H_d \rightarrow H_d / PGL(r+1) =: \overline{P}_{d,g}$$

where $H_d = (\text{Hilb}_{\mathbb{P}^r}^{dt-g+1})^{ss}$, the locus of GIT-semistable points in the Hilbert scheme $\text{Hilb}_{\mathbb{P}^r}^{dt-g+1}$, which is naturally endowed with an action of $PGL(r+1)$ leaving H_d invariant. $\overline{P}_{d,g}$ naturally surjects onto $\overline{\mathcal{M}}_g$ via a proper map $\phi_d : \overline{P}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ such that $\phi_d^{-1}(\overline{\mathcal{M}}_g^0)$ is isomorphic to Pic_g^d .

For $[X] \in \overline{\mathcal{M}}_g$, denote by $\overline{P}_{d,X}$ the inverse image of X by ϕ_d . $\overline{P}_{d,X}$ is a connected projective scheme having at most Δ_X irreducible components, all

of dimension g . In addition, if X is automorphism-free, the smooth locus of $\overline{P}_{d,X}$ is isomorphic to the disjoint union of a finite number of copies of J_X .

Points in H_d correspond to nondegenerate quasistable curves in \mathbb{P}^r embedded by a balanced line bundle.

Let $H_d^s \subseteq H_d$ be the locus of GIT-stable points. These correspond to nondegenerate quasistable curves in \mathbb{P}^r embedded by a stably balanced line bundle of degree d .

Definition 1.4. *Let X be a semistable curve of arithmetic genus $g \geq 2$. We say that X is d -general if all degree d balanced line bundles on X are stably balanced. Otherwise, we will say that X is d -special.*

Denote by $U_d := (\phi_d \circ \pi_d)^{-1}(\overline{M}_g^d)$ the subset of H_d corresponding to d -general curves. U_d is an open subset of H_d where the GIT-quotient is geometric (i.e., all fibers are $PGL(r+1)$ -orbits and all stabilizers are finite and reduced), invariant under the action of $PGL(r+1)$.

$U_d = H_d$ if and only if $(d-g+1, 2g-2) = 1$, so the GIT-quotient yielding $\overline{P}_{d,g}$ is geometric if and only if $(d-g+1, 2g-2) = 1$ (see Prop. 6.2 of loc. cit.).

2. BALANCED PICARD STACKS ON d -GENERAL CURVES

By reasons that will be clear in a moment (see Section 2.1), call $\overline{P}_{d,g}^{Ner}$ the GIT-quotient of U_d by $PGL(r+1)$.

For the time being let

$$G := PGL(r+1).$$

Let us now consider the quotient stack

$$\overline{\mathcal{P}}_{d,g}^{Ner} := [U_d/G].$$

Recall that, given a scheme S over k , a section of $\overline{\mathcal{P}}_{d,g}^{Ner}$ over S consists of a pair $(\phi : E \rightarrow S, \psi : E \rightarrow U_d)$ where ϕ is a G -principal bundle and ψ is a G -equivariant morphism. Arrows correspond to those pullback diagrams which are compatible with the morphism to U_d .

Let $\overline{\mathcal{M}}_g^d \subset \overline{\mathcal{M}}_g$ be the moduli stack of d -general stable curves. There is a natural map from $\overline{\mathcal{P}}_{d,g}^{Ner}$ to $\overline{\mathcal{M}}_g^d$, the restriction to d -general curves of the moduli stack of stable curves, $\overline{\mathcal{M}}_g$. In fact, the restriction to U_d of the stabilization morphism from H_d to $\overline{\mathcal{M}}_g$ factors through $\overline{\mathcal{M}}_g^d$ and, since U_d is invariant under the action of G , this yields a map from $\overline{\mathcal{P}}_{d,g}^{Ner}$ to $\overline{\mathcal{M}}_g^d$.

Recall the following definitions.

Definition 2.1. *We say that a morphism of stacks $f : \mathcal{F} \rightarrow \mathcal{G}$ is representable (resp. strongly representable) if for any scheme Y with a morphism $Y \rightarrow \mathcal{G}$ the fiber product $\mathcal{F} \times_{\mathcal{G}} Y$ is an algebraic space (resp. a scheme).*

Note that morphisms of schemes are always strongly representable.

Definition 2.2. *A coarse moduli space for an stack \mathcal{F} is an algebraic space F together with a morphism $\pi : \mathcal{F} \rightarrow F$ satisfying the following properties:*

- (1) for any algebraically closed field Ω , π induces an isomorphism between the connected components of the groupoids $\mathcal{F}(\text{Spec } \Omega)$ and $F(\text{Spec } \Omega)$;
- (2) π is universal for morphisms from \mathcal{F} onto algebraic spaces.

Example 2.3. GIT-geometric quotients by the action of an algebraic group in a scheme are coarse moduli spaces for the quotient stack associated to that action (see [V89] 2.1 and 2.11).

Lemma 2.4. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a representable morphism of Deligne-Mumford stacks admitting coarse moduli spaces F and G , respectively. Then, if the morphism induced by f in the coarse moduli spaces, $\pi : F \rightarrow G$, is strongly representable, also f is strongly representable.

Proof. We must show that, given a scheme B with a morphism to \mathcal{G} , the fiber product of f with this morphism, \mathcal{F}_B , is a scheme.

$$\begin{array}{ccccc} \mathcal{F}_B & \longrightarrow & \mathcal{F} & \longrightarrow & F \\ \downarrow & & \downarrow f & & \downarrow \pi \\ B & \longrightarrow & \mathcal{G} & \longrightarrow & G \end{array}$$

Since f is representable, we know that \mathcal{F}_B is an algebraic space, so to show that it is indeed a scheme it is enough to show that there is a projective morphism from \mathcal{F}_B to a scheme (see [Vie94] 9.4). Consider the fiber product of the induced morphism from B to G with π , F_B . Since, by hypothesis, π is representable, F_B is a scheme and is endowed with a natural morphism to \mathcal{F}_B , $\rho : \mathcal{F}_B \rightarrow F_B$, the base change over B of the map from \mathcal{F} to F . Since F is the coarse moduli space of \mathcal{F} , this map is proper (see [V89] 2.1), so also ρ is proper. Now, to show that ρ is projective it is enough to see that it has finite fibers, which follows from the fact that the stacks are Deligne-Mumford. \square

Proposition 2.5. The quotient stack $\overline{\mathcal{P}}_{d,g}^{Ner}$ is Deligne-Mumford for every $d \in \mathbb{Z}$ and for every $g \geq 2$ and is strongly representable over $\overline{\mathcal{M}}_g^d$.

Proof. The fact that $\overline{\mathcal{P}}_{d,g}^{Ner}$ is Deligne-Mumford comes from the well known fact that a quotient stack is Deligne-Mumford if and only if the action of the group on the scheme is GIT-geometric, that is, if all stabilizers are finite and reduced. Since U_d is the locus of curves where balanced line bundles are necessarily stably balanced, the Hilbert point of a d -general curve is GIT-semistable if and only if it is GIT-stable, so the GIT-quotient of U_d by G is geometric.

The proof of the strong representability of the natural map from $\overline{\mathcal{P}}_{d,g}^{Ner}$ to $\overline{\mathcal{M}}_g^d$ consists on two steps: first we prove that it is representable and then we use it to prove strong representability.

To prove representability it is sufficient to see that given any section of our quotient stacks over the spectrum of an algebraically closed field k' , the automorphism group of it injects into the automorphism group of its image in $\overline{\mathcal{M}}_g^d$ (see for example [AV02] 4.4.3). But a section of our quotient stack over an algebraically closed field consists of a map onto a orbit of the action of G in U_d . So, the automorphism group of that section is isomorphic to the

stabilizer of the orbit. The image of our section consists of a stable curve X : the stable model of the projective curve associated to that orbit. As this must be d -general, it is GIT-stable and we can use [C94] section 8.2 to conclude that the stabilizer of the orbit injects into the automorphism group of X .

So, the map from $\overline{\mathcal{P}}_{d,g}^{Ner}$ to $\overline{\mathcal{M}}_g^d$ is representable. It follows now immediately that it is also strongly representable from Lemma 2.4 and the fact that the GIT-quotients yielding $\overline{\mathcal{P}}_{d,g}^{Ner}$ and $\overline{\mathcal{M}}_g$ are geometric (see Example 2.3). \square

Definition 2.6. *Let $f : \mathcal{X} \rightarrow S$ be a family of stable curves. A compactification of the relative Picard variety of degree d associated to f is a projective S -scheme P whose fiber over closed points ξ of S is isomorphic to $\phi_d^{-1}(X_\xi)$, where by X_ξ we mean the fiber of f over ξ .*

The following is an immediate consequence of the previous Proposition.

Corollary 2.7. *The Deligne-Mumford stack $\overline{\mathcal{P}}_{d,g}^{Ner}$ gives a functorial way of getting compactifications of the relative Picard variety of degree d for families of d -general curves in the sense of Definition 2.6.*

Remark 2.8. Denote by

$$(2.3) \quad \overline{\mathcal{P}}_{d,g} := [H_d/G]$$

the quotient stack of the action of $G = PGL(r+1)$ in H_d . Then, if $(d - g + 1, 2g - 2) = 1$, $\overline{\mathcal{P}}_{d,g} = \overline{\mathcal{P}}_{d,g}^{Ner}$ and all we said in this section was already proved in [C05] section 5 for $\overline{\mathcal{P}}_{d,g}$.

2.1. Néron models of families of d -general curves. Recall that, given DVR (discrete valuation ring) R with function field K and an abelian variety A_K over K , the Néron model of A_K , $N(A_K)$, is a smooth model of A_K over $B = \text{Spec} R$ defined by the following universal property (cf. [BLR] Definition 1): for every smooth scheme Z over B with a map $u_K : Z_K \rightarrow A_Z$ of its generic fiber, there exists a unique extension of u_K to a B -morphism $u : Z \rightarrow N(A_K)$. Note that $N(A_K)$ may fail to be proper over B but it is always separated.

Let $f : \mathcal{X} \rightarrow B$ be a family of stable curves with \mathcal{X} nonsingular. Denote by X_k the closed fiber of the family and by \mathcal{X}_K its generic fiber. The question is how to construct the Néron model of the Picard variety $\text{Pic}^d \mathcal{X}_K$ in a functorial way over $\overline{\mathcal{M}}_g$. Even if it is natural to look at the Picard scheme (of degree d) of the family, $\text{Pic}_f^d \rightarrow B$, which is smooth and has generic fiber equal to $\text{Pic}^d \mathcal{X}_K$, it turns out to be non satisfactory since it fails to be separated over B if the closed fiber X_k of f is reducible.

Denote by $\mathcal{P}_{d,g}^{Ner}$ the quotient stack $[U_d^{st}/G]$, where U_d^{st} is the locus of points in U_d parametrize d -general stable curves.

It is clear that the statement of Proposition 2.5 holds for $\mathcal{P}_{d,g}^{Ner}$ since U_d^{st} is a G -invariant subscheme of U_d . So, given a family of d -general stable curves $f : \mathcal{X} \rightarrow B$, the fiber product $\mathcal{P}_{d,g} \times_{\overline{\mathcal{M}}_g^d} B$, where $B \rightarrow \overline{\mathcal{M}}_g$ is the moduli map associated to the family f , is a scheme over B , denoted by P_f^d .

$$\begin{array}{ccc}
P_f^d & \longrightarrow & \mathcal{P}_{d,g}^{Ner} \\
\downarrow & & \downarrow \\
B & \longrightarrow & \overline{\mathcal{M}}_g^d
\end{array}$$

Suppose \mathcal{X} is regular. Then, from [C05], Theorem 6.1, we get that $P_f^d \cong N(\text{Pic}^d \mathcal{X}_K)$.

2.2. Combinatorial description of d -general curves in $\overline{\mathcal{M}}_g$. Recall the notions of d -general and d -special curve from Definition 1.4.

Following the notation of [CE], we will denote by Σ_g^d the locus in $\overline{\mathcal{M}}_g$ of d -special curves. So, Σ_g^d consists of stable curves X of genus g such that $\tilde{B}_X^d \setminus B_X^d \neq \emptyset$ (see Definition 1.1). In particular, Σ_g^d is contained in the closed subset of $\overline{\mathcal{M}}_g$ consisting of reducible curves. Let us also denote by $\overline{\mathcal{M}}_g^d$ the locus of d -general genus g stable curves (so $\Sigma_g^d \cup \overline{\mathcal{M}}_g^d = \overline{\mathcal{M}}_g$, for all $d \in \mathbb{Z}$).

From [C94], Lemma 6.1, we know that $\overline{\mathcal{M}}_g^d$ is the image under ϕ_d of U_d , so it is an open subset of $\overline{\mathcal{M}}_g$.

Recall that a *vine curve* is a curve with two smooth irreducible components meeting in an arbitrary number of nodes. The closure in $\overline{\mathcal{M}}_g$ of the vine curves of genus g is precisely the locus of reducible curves.

In Proposition 2.13, we give a geometric description of Σ_g^d .

Example 2.9. Let $d = 1$. From [CE], Prop. 3.15 we know that, if g is odd, Σ_g^1 is empty and that if g is even, Σ_g^1 is the closure in $\overline{\mathcal{M}}_g$ of the locus of curves $X = C_1 \cup C_2$, with C_1 and C_2 smooth of the same genus and $\sharp(C_1 \cap C_2) = k$ odd.

Observe that, from the above example, we get that Σ_g^1 is the closure in $\overline{\mathcal{M}}_g$ of the 1-special vine curves of genus g . In what follows we will see that this is always the case for any degree.

Lemma 2.10. *Let d be an integer greater or equal to 1. Then Σ_g^d is the closure in $\overline{\mathcal{M}}_g$ of the locus of d -special vine curves.*

Proof. Let X be a genus g d -special curve. As X is stable, using Remark 1.3 (C), this means that there is a connected proper subcurve Z of X and a balanced line bundle L on X such that $\deg_Z L = m_Z(d)$.

So, let Z be a connected proper subcurve of X such that $\deg_Z L = m_Z(d)$ and such that w_Z is maximal among the subcurves satisfying this relation. The complementary curve of Z in X , Z' must be such that

$$\deg_{Z'} L = d - \deg_Z L = d - m_Z(d) = M_{Z'}(d).$$

Let us see that Z' is connected as well.

By contradiction, suppose $Z' = Z'_1 \cup \dots \cup Z'_s$ is a union of connected components with $s > 1$. As $\deg_{Z'} L = M_{Z'}(d)$, also each one of its connected components Z'_i , $i = 1, \dots, s$, must be such that $\deg_{Z'_i} L = M_{Z'_i}(d)$. In fact,

suppose one of them, say Z'_j , is such that $\deg_{Z'_j} L < M_{Z'_j}(d)$. Then,

$$\deg_{Z'} L = \sum_{i=1}^s \deg_{Z'_i} L < \sum_{i=1}^s \frac{dw_{Z'_i}}{2g-2} + \frac{k_{Z'_i}}{2} = \frac{dw_{Z'}}{2g-2} + \frac{k_{Z'}}{2} = M_{Z'}(d)$$

leading us to a contradiction. Note that the sum of the $k_{Z'_i}$'s is $k_{Z'}$ because, being the Z'_i 's the connected components of Z' , they do not meet each other.

Now, let us consider $W := Z \cup Z'_1$. As $s > 1$, W is a connected proper subcurve of X with $w_W = w_Z + w_{Z'_1} > w_Z$. Indeed, as X is stable,

$$(2.4) \quad 0 < w_Y < 2g - 2$$

for every proper subcurve Y of X , since there are no exceptional components. Moreover,

$$\deg_W L = \deg_Z L + \deg_{Z'_1} L = \frac{dw_Z}{2g-2} - \frac{k_Z}{2} + \frac{dw_{Z'_1}}{2g-2} + \frac{k_{Z'_1}}{2} = \frac{dw_W}{2g-2} - \frac{k_W}{2}$$

because, being Z'_1 a connected component of Z' , we have that

$$k_Z - k_{Z'_1} = k_Z - \#(Z \cap Z'_1) = k_W.$$

So, W is a connected proper subcurve of X with $\deg_W L = m_W(d)$ and with $w_W > w_Z$. This way, we achieved a contradiction by supposing that Z' is not connected.

As both Z and Z' are limits of smooth curves and Σ_g^d is closed in \overline{M}_g , then X lies in the closure in \overline{M}_g of the locus of genus g d -special vine curves. \square

Given integers d and g , we will use the following notation to indicate greatest common divisor

$$G_d := (d - g + 1, 2g - 2).$$

From [C94], we know that Σ_g^d is a proper closed subset of \overline{M}_g and that $\Sigma_g^d = \emptyset$ if and only if $G_d = 1$ (see Prop. 6.2 of loc. cit.).

Remark 2.11. From Lemma 2.10 and Remark 1.3(D) we conclude that a stable curve X is d -special if and only if there is a connected proper subcurve Z of X such that $\overline{X \setminus Z}$ is connected and $\frac{2g-2}{G_d}$ divides w_Z .

Remark 2.12. If $G_d = 2g - 2$, which means that $d \equiv (g - 1) \pmod{2g - 2}$, an immediate consequence of the previous Remark is that all reducible curves are $(g - 1)$ -special. This is the opposite situation to the case $G_d = 1$.

From Remark 2.11 we see that Σ_g^d depends only on G_d . This is evident in the following proposition, where we give a geometric description of Σ_g^d .

Proposition 2.13. *Let d be an integer greater or equal to 1. Then Σ_g^d is the closure in \overline{M}_g of vine curves $X = C_1 \cup C_2$ such that*

$$\frac{2g-2}{G_d} \mid w_{C_1}.$$

More precisely, Σ_g^d is the closure in \overline{M}_g of the following vine curves: given integers m and k with

$$1 \leq m < G_d$$

and

$$1 \leq k \leq \min\left\{\frac{2g-2}{G_d}m + 2, 2g - \frac{2g-2}{G_d}m\right\}, \quad k \equiv \frac{2g-2}{G_d}m \pmod{2},$$

then $X = C_1 \cup C_2$, with $\sharp(C_1 \cap C_2) = k$, and

- $g(C_1) = \frac{g-1}{G_d}m - \frac{k}{2} + 1$;
- $g(C_2) = g - \frac{g-1}{G_d} - \frac{k}{2}$.

Proof. The first part of the proposition is an immediate consequence of Lemma 2.10 and Remark 2.11.

Now, let X be a d -special genus g vine curve $X = C_1 \cup C_2$ with $\sharp(C_1 \cap C_2) = k$. From Remark 2.11 we know that there exists an integer m such that

$$m \frac{2g-2}{G_d} = w_{C_1}$$

with $1 \leq m < G_d$ because, as X is a stable curve, $\frac{w_{C_1}}{2g-2}$ must be smaller than 1.

As $w_{C_1} = 2g(C_1) - 2 + k$, we get that $k \equiv \frac{2g-2}{G_d}m \pmod{2}$ and that

$$g(C_1) = \frac{g-1}{G_d}m - \frac{k}{2} + 1.$$

Now, as $g = g(C_1) - g(C_2) + k - 1$, we get that

$$g(C_2) = g - \frac{g-1}{G_d} - \frac{k}{2}.$$

As $g(C_1)$ and $g(C_2)$ must be greater or equal than 0, we get, respectively, that

$$k \leq \frac{2g-2}{G_d}m + 2 \text{ and } k \leq 2g - \frac{2g-2}{G_d}m.$$

It is easy to see that if $g(C_1)$ or $g(C_2)$ are equal to 0 then $k \geq 3$. So, the vine curves we constructed are all stable. \square

Remark 2.14. Since by smoothing a vine curve in any of its nodes we get an irreducible curve, we see that the above set of “generators” of Σ_g^d is minimal in the sense that none of them lies in the closure of the others.

The dependence of Σ_g^d on G_d gets even more evident in the following proposition.

Proposition 2.15. *For every $d, d' \in \mathbb{Z}$, $G_d | G_{d'}$ if and only if $\Sigma_g^d \subset \Sigma_g^{d'}$.*

Proof. That $G_d | G_{d'}$ implies that $\Sigma_g^d \subset \Sigma_g^{d'}$ is immediate from Remark 2.11.

Now, suppose $\Sigma_g^d \subset \Sigma_g^{d'}$. If $G_d = 1$ then obviously $G_d | G_{d'}$. For $G_d \neq 1$ we will conclude by contradiction that $G_d | G_{d'}$. So, suppose $G_{d'} \nmid G_d$. Then, also $\frac{2g-2}{G_{d'}} \nmid \frac{2g-2}{G_d}$. We will show that there exists a stable curve X consisting of two smooth irreducible components C_1 and C_2 meeting in δ nodes ($\delta \geq 1$) which is d -special but not d' -special.

Take X such that $w_{C_1} = \frac{2g-2}{G_d}$. If such a curve exists and is stable then we are done because X will clearly be d -special and not d' -special. In fact, by construction, $\frac{2g-2}{G_{d'}}$ does not divide w_{C_1} and $\frac{2g-2}{G_{d'}}$ will not divide w_{C_2} too because $w_{C_2} = (2g-2) - \frac{2g-2}{G_d}$.

So, X must be such that

- $g(C_1) = \frac{g-1}{G_d} + 1 - \frac{\delta}{2}$
- $g(C_2) = g - \frac{g-1}{G_d} - \frac{\delta}{2}$
- $\delta \geq 1$ and $\delta \equiv \frac{2g-2}{G_d} \pmod{2}$.

As $g(C_i)$ must be greater or equal than 0 and the curve X must be stable, we must check if such a construction is possible.

So, if $\frac{2g-2}{G_d} \equiv 1 \pmod{2}$, take $\delta = 1$. Then we will have that $g(C_1) = \frac{g-1}{G_d} + \frac{1}{2}$ and $g(C_2) = g - \frac{g-1}{G_d} - \frac{1}{2}$, which are both greater than 1 because we are considering $G_d > 1$.

If $\frac{2g-2}{G_d} \equiv 0 \pmod{2}$, take $\delta = 2$. Then we will have that $g(C_1) = \frac{g-1}{G_d}$ and $g(C_2) = g - \frac{g-1}{G_d} - 1$, again both greater than 1. We conclude that X is a stable curve. \square

The following is immediate.

Corollary 2.16. *For all d and d' , $\Sigma_g^d = \Sigma_g^{d'}$ if and only if $G_d = G_{d'}$.*

For each positive divisor M of $2g - 2$ there is an integer $d = M + g - 1$ such that $G_d = M$. So, for each such M , we can define

$$\Sigma_{g,M} := \Sigma_g^d \text{ and } \overline{M}_g^M = \overline{M}_g \setminus \Sigma_{g,M}.$$

For example, \overline{M}_g^{2g-2} consists of irreducible curves and $\overline{M}_g^1 = \overline{M}_g$.

The following is now immediate.

Proposition 2.17. *The open subsets \overline{M}_g^M associated to the positive divisors M of $2g - 2$, form a lattice of open subschemes of \overline{M}_g such that $\overline{M}_g^M \subset \overline{M}_g^{M'}$ if and only if $M' | M$.*

3. MODULAR DESCRIPTION OF BALANCED PICARD STACKS OVER \overline{M}_g

Suppose $(d - g + 1, 2g - 2) = 1$. Then $\overline{\mathcal{M}}_g^d = \overline{\mathcal{M}}_g$ and $\overline{\mathcal{P}}_{d,g}^{Ner} = \overline{\mathcal{P}}_{d,g}$ (see section 2.2). Moreover, from [C05], 5.10, we know that $\overline{\mathcal{P}}_{d,g}$ is the “rigidification” in the sense of [ACV01] (see section 4 below) of the category whose sections over a scheme S are pairs $(f : \mathcal{X} \rightarrow S, \mathcal{L})$ where f is a family of quasistable curves of genus g and \mathcal{L} is a balanced line bundle on \mathcal{X} of relative degree d . Arrows between such pairs are given by cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ f \downarrow & & \downarrow f' \\ S & \longrightarrow & S' \end{array}$$

and an isomorphism $\mathcal{L} \cong h^* \mathcal{L}' \otimes f^* M$, for some $M \in \text{Pic } S$.

This description uses heavily the existence of Poincaré line bundles for families of quasistable curves, established in loc. cit., Lemma 5.5. However, this works only if $(d - g + 1, 2g - 2) = 1$.

In order to overcome this difficulty we will try to define the stack of line bundles of families of stable curves.

We will start by recalling the definition of “Picard stack associated to a morphism of schemes”. Roughly speaking, a *Picard stack* is a stack together

with an “addition” operation which is both associative and commutative. The theory of Picard stacks is developed by Deligne and Grothendieck on Section 1.4 of Exposé XVIII in [SGA4]. We will not include here the precise definition but we address the reader to [ibid.], [L-MB00] 14.4 and [BF], section 2.

Given a scheme X over S with structural morphism $f : X \rightarrow S$, the S -stack of (quasi-coherent) invertible \mathcal{O}_X -modules, $\mathcal{P}ic_{X/S}$, is a Picard stack: the one associated to the complex of length one

$$\tau_{\leq 0}(Rf_*\mathbb{G}_m[1]).$$

So, given an S -scheme T , $\mathcal{P}ic_{X/S}(T)$ is the groupoid whose objects are invertible \mathcal{O}_{X_T} -modules and whose morphisms are the isomorphisms between them (notation as in 1.1.2).

$\mathcal{P}ic_{X/S}$ fits in the exact sequence below, where, given an S -scheme T , $\mathcal{P}ic_{X/S}(T)$ is defined as $\text{Pic } X_T/f_T^*(\text{Pic } T)$ and $B\mathbb{G}_m(T)$ is the group of line bundles over T .

$$0 \rightarrow B\mathbb{G}_m \rightarrow \mathcal{P}ic_{X/S} \rightarrow \mathcal{P}ic_{X/S} \rightarrow 0$$

Now, let us consider the forgetful morphism of stacks $\pi : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$. The morphism π is strongly representable since, given a morphism Y with a map $h : Y \rightarrow \overline{\mathcal{M}}_g$, the fiber product $Y \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$ is isomorphic to the image of Id_Y under h , which is a family of stable curves of genus g , say $\mathcal{C} \rightarrow Y$.

So, we define the category $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$ associated to π as follows. Given a scheme Y , morphisms from Y to $\overline{\mathcal{M}}_g$ correspond to families of stable curves over Y . So, the objects of $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}(Y)$ are given by pairs $(\mathcal{C} \rightarrow Y, \mathcal{L})$ where $\mathcal{C} \rightarrow Y$ is the family of stable curves of genus g associated to a map $Y \rightarrow \overline{\mathcal{M}}_g$ and \mathcal{L} is a line bundle on $\mathcal{C} \cong Y \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$. Morphisms between two such pairs are given by cartesian diagrams

$$(3.5) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{h} & \mathcal{C}' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

together with an isomorphism $\mathcal{L} \cong h^*\mathcal{L}'$.

We will now concentrate on the following full subcategory of $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$ (and on a compactification of it).

Definition 3.1. Let $\mathcal{G}_{d,g}$ (respectively $\overline{\mathcal{G}}_{d,g}$) be the category whose objects are pairs $(f : \mathcal{C} \rightarrow Y, \mathcal{L})$ where f is a family of stable (respectively quasistable) curves of genus g and \mathcal{L} a balanced line bundle of relative degree d over Y . Morphisms between two such pairs are defined as in $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$.

The aim of the present section is to show that both $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$ are algebraic (Artin) stacks. We will do it by directly showing that they are isomorphic to the quotient stacks we are about to define.

Let

$$G := GL(r+1).$$

Recall from the section before that G acts on H_d , the locus of GIT-semistable points in $\text{Hilb}_{\mathbb{P}^r}^{dt-g+1}$, by projecting onto $PGL(r+1)$.

Consider also the open subset of H_d parametrize points corresponding to stable curves and denote it by H_d^{st} . It is easy to see that H_d^{st} is a G -equivariant subset of H_d . So, we can consider the quotient stacks $[H_d^{st}/G]$ and $[H_d/G]$. Given a scheme S , $[H_d^{st}/G](S)$ (respectively $[H_d/G](S)$) consists of G -principal bundles $\phi : E \rightarrow S$ with a G -equivariant morphism $\psi : E \rightarrow H_d$ (respectively $\psi : E \rightarrow H_d^{st}$). Morphisms are given by pullback diagrams which are compatible with the morphism to H_d (resp. H_d^{st}).

Theorem 3.2. *The quotient stacks $[H_d^{st}/GL(r+1)]$ and $[H_d/GL(r+1)]$ are isomorphic respectively to $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$.*

Proof. Since the proof is the same for both cases we will do it only for $[H_d/GL(r+1)]$.

We must show that, for every scheme $S \in SCH_k$, the groupoids $\overline{\mathcal{G}}_{d,g}(S)$ and $[H_d/G](S)$ are equivalent.

Let $(f : \mathcal{X} \rightarrow S, \mathcal{L})$ be a pair consisting of a family f of quasistable curves and a balanced line bundle \mathcal{L} of relative degree d on \mathcal{X} . We must produce a principal G -bundle E on S and a G -equivariant morphism $\psi : E \rightarrow H_d$. Since we can take d very large with respect to g (see Remark 1.3 (E)), we may assume that $f_*(\mathcal{L})$ is locally free of rank $r+1 = d - g + 1$. Then, the frame bundle of $f_*(\mathcal{L})$ is a principal $GL(r+1)$ -bundle: call it E . Now, to find the G -equivariant morphism to H_d , consider the family $\mathcal{X}_E := \mathcal{X} \times_S E$ polarized by \mathcal{L}_E , the pullback of \mathcal{L} to \mathcal{X}_E . \mathcal{X}_E is a family of quasistable curves of genus g and \mathcal{L}_E is balanced and relatively very ample. By definition of frame bundle, $f_{E*}(\mathcal{L}_E)$ is isomorphic to $\mathbb{C}^{(r+1)} \times E$, so that \mathcal{L}_E gives an embedding over E of \mathcal{X}_E in $\mathbb{P}^r \times E$. By the universal property of the Hilbert scheme H , this family determines a map $\psi : E \rightarrow H_d$. It follows immediately that ψ is a G -equivariant map.

Let us check that isomorphisms in $\overline{\mathcal{G}}_{d,g}(S)$ leads canonically to isomorphisms in $[H_d/G](S)$.

An isomorphism between two pairs $(f : \mathcal{X} \rightarrow S, \mathcal{L})$ and $(f' : \mathcal{X}' \rightarrow S, \mathcal{L}')$ consists of an isomorphism $h : \mathcal{X} \rightarrow \mathcal{X}'$ over S and an isomorphism of line bundles $\mathcal{L} \cong h^*\mathcal{L}'$.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ & \searrow f & \swarrow f' \\ & S & \end{array}$$

These determine an unique isomorphism between $f_*(\mathcal{L})$ and $f'_*(\mathcal{L}')$ as follows

$$f_*(\mathcal{L}) \cong f_*(h^*\mathcal{L}') \cong f'_*(h_*(h^*\mathcal{L}')) \cong f'_*(\mathcal{L}').$$

As taking the frame bundle gives an equivalence between the category of vector bundles of rank $r+1$ over S and the category of principal $GL(r+1)$ -bundles over S , the isomorphism $f_*(\mathcal{L}) \cong f'_*(\mathcal{L}')$ leads to an unique isomorphism between their frame bundles, call them E and E' respectively. This isomorphism must be compatible with the G -equivariant morphisms $\psi : E \rightarrow H_d$ and $\psi' : E' \rightarrow H_d$ because they are determined by the induced curves \mathcal{X}_E and $\mathcal{X}'_{E'}$ embedded in \mathbb{P}^r by \mathcal{L}_E and $\mathcal{L}'_{E'}$.

Conversely, given a section $(\phi : E \rightarrow S, \psi : E \rightarrow H_d)$ of $[H_d/G]$ over S , let us construct a family of quasistable curves of genus g over S and a balanced line bundle of relative degree d on it.

Let \mathcal{C}_d be the restriction to H_d of the universal family on $\text{Hilb}_{\mathbb{P}^r}^{dt-g+1}$. The pullback of \mathcal{C}_d by ψ gives a family \mathcal{C}_E on E of quasistable curves of genus g and a balanced line bundle \mathcal{L}_E on \mathcal{C}_E which embeds \mathcal{C}_E as a family of curves in \mathbb{P}^r . As ψ is G -invariant and ϕ is a G -bundle, the family \mathcal{C}_E descends to a family \mathcal{C}_S over S , where $\mathcal{C}_S = \mathcal{C}_E/G$. In fact, since \mathcal{C}_E is flat over E and E is faithfully flat over S , \mathcal{C}_S is flat over S too (see [EGA4], Prop. 2.5.1).

Now, since the action of G on \mathcal{C}_d is naturally linearized (see [C94], 1.4), also the action of G on E can be linearized to an action on \mathcal{L}_E , yielding descent data for \mathcal{L}_E ([SGA1], Proposition 7.8). Moreover, \mathcal{L}_E is relatively (very) ample so, using the fact that ϕ is a principal G -bundle, we conclude that \mathcal{L}_E descends to a relatively very ample balanced line bundle on \mathcal{C}_S , \mathcal{L}_S (see proof of Proposition 7.1 in [GIT]).

It is straightforward to check that an isomorphism on $[H_d/G](S)$ leads to an unique isomorphism in $\overline{\mathcal{G}}_{d,g}(S)$. \square

We will call $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$ respectively *balanced Picard stack* and *compactified balanced Picard stack*. The relation between $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$ and the stacks $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ defined in [C05] will be clear in the following section.

Remark 3.3. Since \mathbb{G}_m is always included in the stabilizers at every point of the action of G both in H_d and in H_d^{st} , the quotient stacks above are never Deligne-Mumford. However, they are, of course Artin stacks with a presentation given by the schemes H_d^{st} and H_d , respectively.

Notice also that, since the scheme H_d is nonsingular and closed (see Lemma 2.2 in [C94]), the algebraic stack $\overline{\mathcal{G}}_{d,g}$ is a smooth compactification of $\mathcal{G}_{d,g}$.

Let

$$d\mathcal{G}_{d,g}$$

be the category over $SC H_k$ whose sections over a scheme S , $d\mathcal{G}_{d,g}(S)$, consists of pairs $(f : \mathcal{X} \rightarrow S, \mathcal{L})$, where f is a family of d -general quasistable curves of genus g and \mathcal{L} is an S -flat balanced line bundle on \mathcal{X} of relative degree d . Arrows between two such pairs are given by cartesian diagrams like in (3.5).

Using the same proof of Prop. 3.2 we conclude that $d\mathcal{G}_{d,g}$ is isomorphic to the quotient stack $[U_d/G]$.

4. RIGIDIFIED BALANCED PICARD STACKS

Recall that in (2.3) we denoted the quotient stack $[H_d/PGL(r+1)]$ by $\overline{\mathcal{P}}_{d,g}$. Define analogously

$$\mathcal{P}_{d,g} := [H_d^{st}/PGL(r+1)]$$

(recall that $H_d^{st} \subset H_d$ parametrizes embedded stable curves).

In what follows we will relate $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$, respectively, with $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ using the notion of rigidification of a stack along a group scheme defined by Abramovich, Vistoli and Corti in [ACV01], 5.1.

Note that each object $(f : \mathcal{X} \rightarrow S, \mathcal{L})$ in $\overline{\mathcal{G}}_{d,g}$ have automorphisms given by scalar multiplication by an element of $\Gamma(\mathcal{X}, \mathbb{G}_m)$ along the fiber of \mathcal{L} . Since these automorphisms fix \mathcal{X} , there is no hope that our stack $\overline{\mathcal{G}}_{d,g}$ can be representable over $\overline{\mathcal{M}}_g$ (see [AV02], 4.4.3). The rigidification procedure removes those automorphisms.

More precisely, the set up of rigidification consists of:

- a stack \mathcal{G} over a base scheme \mathbb{S} ;
- a finitely presented group scheme G over \mathbb{S} ;
- for any object ξ of \mathcal{G} over an \mathbb{S} -scheme S , an embedding

$$i_\xi : G(S) \rightarrow \text{Aut}_S(\xi)$$

compatible with pullbacks.

Then the statement (Theorem 5.1.5 in [ACV01]) is that there exists a stack \mathcal{G}^G and a morphism of stacks $\mathcal{G} \rightarrow \mathcal{G}^G$ over \mathbb{S} satisfying the following conditions:

- (1) For any object $\xi \in \mathcal{G}(S)$ with image $\eta \in \mathcal{G}^G(S)$, the set $G(S)$ lies in the kernel of $\text{Aut}_S(\xi) \rightarrow \text{Aut}_S(\eta)$;
- (2) The morphism $\mathcal{G} \rightarrow \mathcal{G}^G$ above is universal for morphisms of stacks $\mathcal{G} \rightarrow \mathcal{F}$ satisfying condition (1) above;
- (3) If S is the spectrum of an algebraically closed field, we have in (1) above that $\text{Aut}_S(\eta) = \text{Aut}_S(\xi)/G(S)$;
- (4) A moduli space for \mathcal{G} is also a moduli space for \mathcal{G}^G .

\mathcal{G}^G is the *rigidification* of \mathcal{G} along G .

By taking $\mathbb{S} = \text{Spec } k$, $\mathcal{G} = \overline{\mathcal{G}}_{d,g}$ and $G = \mathbb{G}_m$ we see that our situation fits up in the setting above. It is easy to see that, since $GL(r+1)$ acts on H_d by projection onto $PGL(r+1)$, $[H_d/PGL(r+1)]$ is the rigidification of $[H_d/GL(r+1)] \cong \overline{\mathcal{G}}_{d,g}$ along \mathbb{G}_m : denote it by $\overline{\mathcal{G}}_{d,g}^{\mathbb{G}_m}$. Naturally, the same holds for $[H_d^{st}/PGL(r+1)]$ and $\mathcal{G}_{d,g}^{\mathbb{G}_m}$. The following is now immediate.

Proposition 4.1. *The stacks $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are isomorphic, respectively, to $\mathcal{G}_{d,g}^{\mathbb{G}_m}$ and $\overline{\mathcal{G}}_{d,g}^{\mathbb{G}_m}$. In particular, $\mathcal{G}_{d,g}$ and $\overline{\mathcal{G}}_{d,g}$ are \mathbb{G}_m -gerbes over $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$, respectively.*

Remark 4.2. The previous proposition holds, of course, also for the rigidification along \mathbb{G}_m of $d\mathcal{G}_{d,g}$, $d\mathcal{G}_{d,g}^{\mathbb{G}_m}$, and $\overline{\mathcal{P}}_{d,g}^{Ner}$, which we have studied in section 2 (see the definition of $d\mathcal{G}_{d,g}$ in the end of section 3).

Recall from the beginning of section 3 that, if $(d - g + 1, 2g - 2) = 1$, $\overline{\mathcal{P}}_{d,g}$ has a modular description as the rigidification for the action of \mathbb{G}_m in a certain category.

In order to remove from $\overline{\mathcal{G}}_{d,g}$ the automorphisms given by \mathbb{G}_m , we first consider the auxiliary category $\overline{\mathcal{A}}_{d,g}$, whose objects are the same of $\overline{\mathcal{G}}_{d,g}$ but where morphisms between pairs $(\mathcal{C} \rightarrow Y, \mathcal{L})$ and $(\mathcal{C}' \rightarrow Y', \mathcal{L}')$ are given by equivalence classes of morphisms in $\overline{\mathcal{G}}_{d,g}$ by the following relation. Given a

cartesian diagram

$$(4.6) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{h} & \mathcal{C}' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

and isomorphisms $\phi : \mathcal{L} \rightarrow h^*\mathcal{L}'$ and $\psi : \mathcal{L} \rightarrow h^*\mathcal{L}'$, we say that ϕ is equivalent to ψ if there exists $\alpha \in \mathbb{G}_m$ such that $\underline{\alpha} \circ \psi = \phi$, where by $\underline{\alpha}$ we mean the morphism induced by α in \mathcal{L}' (fiberwise multiplication by α).

There is an obvious morphism of $\overline{\mathcal{G}}_{d,g} \rightarrow \overline{\mathcal{A}}_{d,g}$ satisfying property (1) above and universal for morphisms of $\overline{\mathcal{G}}_{d,g}$ in categories satisfying it. However, it turns out that $\overline{\mathcal{A}}_{d,g}$ is not a stack. In fact, it is not even a prestack since, given an étale cover $\{\coprod_i Y_i \rightarrow Y\}$ of Y , the natural morphism $\overline{\mathcal{A}}_{d,g}(Y)$ to $\overline{\mathcal{A}}_{d,g}(\coprod_i Y_i \rightarrow Y)$, the category of effective descent data for this covering, is not fullyfaithful but just faithful.

Let us now consider the category $\overline{\mathcal{C}}_{d,g}$, with the following modular description. A section of $\overline{\mathcal{C}}_{d,g}$ over a scheme S is given by a pair $(f : \mathcal{X} \rightarrow S, \mathcal{L})$, where f is a family of quasistable curves of genus g and \mathcal{L} is a balanced line bundle on \mathcal{X} of relative degree d . Arrows between such pairs are given by cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ f \downarrow & & \downarrow f' \\ S & \longrightarrow & S' \end{array}$$

and equivalence classes of isomorphisms $\mathcal{L} \cong h^*\mathcal{L}' \otimes f^*M$, for some $M \in \text{Pic } S$, for the following relation. Isomorphisms $\phi : \mathcal{L} \rightarrow h^*\mathcal{L}' \otimes f^*M$ and $\psi : \mathcal{L} \rightarrow h^*\mathcal{L}' \otimes f^*N$ are equivalent if there exists an isomorphism $g : N \rightarrow M$ of line bundles on S such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\phi} & h^*\mathcal{L}' \otimes f^*M \\ & \searrow \psi & \uparrow \text{id} \otimes f^*g \\ & & h^*\mathcal{L}' \otimes f^*N \end{array}$$

Straightforward computations show that $\overline{\mathcal{C}}_{d,g}$ is a prestack. Moreover, given the étale cover $(\coprod_i Y_i \rightarrow Y)$ of Y , $\overline{\mathcal{A}}_{d,g}(\coprod_i Y_i \rightarrow Y)$ is isomorphic to $\overline{\mathcal{C}}_{d,g}(\coprod_i Y_i \rightarrow Y)$.

So, we conclude that the stackification of $\overline{\mathcal{C}}_{d,g}$ is the rigidification of $\overline{\mathcal{G}}_{d,g}$ under the action of \mathbb{G}_m .

Proposition 4.3. *The stack $\overline{\mathcal{P}}_{d,g}$ (resp. $\mathcal{P}_{d,g}$) is the stackification of the prestack whose sections over a scheme S are given by pairs $(f : \mathcal{X} \rightarrow S, \mathcal{L})$, where f is a family of quasistable (resp. stable) curves of genus g and \mathcal{L} is a balanced line bundle on \mathcal{X} of relative degree d . Arrows between two such*

pairs are given by cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ f \downarrow & & \downarrow f' \\ S & \longrightarrow & S' \end{array}$$

and an isomorphism $\mathcal{L} \cong h^* \mathcal{L}' \otimes f^* M$, for some $M \in \text{Pic } S$.

Remark 4.4. Let $d \gg 0$. Then, as in the case $(d - g + 1, 2g - 2) = 1$, there is a canonical map from $\overline{\mathcal{P}}_{d,g}$ and $\mathcal{P}_{d,g}$ for $\overline{P}_{d,g}$ and $P_{d,g}$, the GIT-quotients of H_d and H_d^{st} , respectively, by the action of $PGL(r + 1)$. If $(d - g + 1, 2g - 2) \neq 1$, these quotients are not geometric, which implies that these maps are universally closed but not separated. So, $\overline{\mathcal{P}}_{d,g}$ and $\mathcal{P}_{d,g}$ are not coarse moduli spaces for those stacks since the associated maps from the stacks onto them are not proper. However, at least if the base field has characteristic 0, we have that the GIT-quotients are *good moduli spaces* in the sense of Alper (see [A08]).

4.1. Rigidified balanced Picard stacks and Néron models. The main question now is the following: does our balanced Picard stack $\mathcal{P}_{d,g}$ parametrizes Néron models of families of stable curves for every d as $\mathcal{P}_{d,g}^{Ner}$ does for families of d -general curves (see 2.1)?

Given a family of stable curves $f : \mathcal{X} \rightarrow B = \text{Spec } R$, we will denote by Q_f^d the base-change of the map $\mathcal{P}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ by the natural map $B \rightarrow \overline{\mathcal{M}}_g$: $B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g}$. Is Q_f^d isomorphic to $N(\text{Pic}^d \mathcal{X}_K)$?

The problem here is that the map $\mathcal{P}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ is not representable in general. In fact, if it were, $\mathcal{P}_{d,g}$ would be a Deligne-Mumford stack. Indeed, it is easy to see that a stack with a representable map to a Deligne-Mumford stack is necessarily Deligne-Mumford. As we already mentioned, $\mathcal{P}_{d,g}$ is not Deligne-Mumford in general since it is the quotient stack associated to a non-geometric GIT-quotient.

As a consequence of this, if the closed fiber of f is not d -general, then Q_f^d is not even equivalent to an algebraic space. In fact, from a common criterion for representability (see for example [AV02], 4.4.3 or the proof of Proposition 2.5), we know that Q_f^d would be equivalent to an algebraic space if and only if the automorphism group of every section of $\mathcal{P}_{d,g}$ over k with image in $\overline{\mathcal{M}}_g$ isomorphic to $X_k \rightarrow k$ injects into the automorphism group of X_k . Since such a section corresponds to a map onto its orbit in H_d by the action of $PGL(r + 1)$, the automorphism group of such a section is isomorphic to the stabilizer of that orbit. So, as X_k is not d -general, the stabilizer of its associated orbit in H_d is not finite, which implies that it cannot have an injective morphism to the automorphism group of X_k , which is, of course, finite.

The following example will clarify what we just said.

Example 4.5. Let $d = g - 1$ and $f : \mathcal{X} \rightarrow B = \text{Spec } R$ be a family of stable curves such that \mathcal{X} is regular and X_k is a reducible curve consisting of two smooth components C_1 and C_2 of genus g_1 and g_2 respectively, meeting in one point (of course, $g = g_1 + g_2$). Then, the fiber over k of Q_f^d is

a stack with a presentation given by a subscheme of the Hilbert scheme H_d , consisting of two connected components of dimension $r(r+2)+g$ (see [C94] Example 7.2). These correspond to projective realizations of X_k on \mathbb{P}^r given by line bundles with the two possible balanced multidegrees on X_k : $(g_1, g_2 - 1)$ and $(g_1 - 1, g_2)$. By Proposition 5.1 of [C94], given a point h in one of these components, there is a point in $\overline{O_{PGL(r+1)}(h)}$ representing the quasistable curve with stable model X_k embedded by a line bundle of multidegree $(g_1 - 1, 1, g_2 - 1)$.

So, as a stack, the fiber of Q_f^d over k is reducible but the GIT quotient of the Hilbert scheme presenting it by the action of $PGL(r+1)$ is irreducible and isomorphic to the Jacobian of X_k . As a consequence, Q_f^d can never be isomorphic to the Néron model $N(\text{Pic}^d \mathcal{X}_K)$.

This is an example of a situation where the GIT-quotient identifies two components of the Hilbert scheme while in the quotient stack these two components remain separated.

So, we have.

Proposition 4.6. *Let $f : \mathcal{X} \rightarrow B = \text{Spec } R$ be a family of stable curves with \mathcal{X} regular. Then, using the notation above, $Q_f^d \cong N(\text{Pic}^d \mathcal{X}_K)$ if and only if and only if X_k is a d -general curve.*

4.1.1. *Functoriality for non d -general curves.* Let $f : \mathcal{X} \rightarrow S$ be a family of stable curves. Denote by \overline{P}_f^d the fiber product of $\overline{\mathcal{P}}_{d,g}$ by the moduli map of f , $\mu_f : S \rightarrow \overline{\mathcal{M}}_g$.

Recall that, if $(d-g+1, 2g-2) = 1$, \overline{P}_f^d is a compactification of the relative degree d Picard variety associated to f in the sense of 2.6 (see Remark 2.8).

Let now $(d-g+1, 2g-2) \neq 1$. Then, since $\overline{\mathcal{P}}_{d,g}$ is not representable over $\overline{\mathcal{M}}_g$, we have just observed that the same cannot be true in general.

However, we have the following result.

Proposition 4.7. *Notation as before. Then \overline{P}_f^d has a canonical proper map onto a compactification P of the relative degree d Picard variety associated to f .*

Proof. If all fibers of f are d -general, then from Corollary 2.7 it follows that \overline{P}_f^d is a scheme and it gives a compactification of the relative degree d Picard variety associated to f .

Suppose not all fibers of f are d -general. Then \overline{P}_f^d is a stack with a presentation given by the subscheme H_d^f of H_d corresponding to the closure in H_d of the locus parametrize curves isomorphic to the fibers of f . H_d^f is naturally invariant for the action of $PGL(r+1)$ on it. Let P be the GIT-quotient of H_d^f by $PGL(r+1)$. P gives a compactification of the relative degree d Picard variety associated to f . The proper map $H_d^f \rightarrow P$ factorizes through $H_d^f \rightarrow \overline{P}_f^d$, the presentation map. In fact, even if P is not a coarse moduli space for \overline{P}_f^d , there is a canonical map from \overline{P}_f^d onto P which is universal for morphisms of \overline{P}_f^d into schemes (see [V89] section 2). Now, since the map $H_d^f \rightarrow P$ is proper, then $\overline{P}_f^d \rightarrow P$ must be proper as well.

□

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